Matched Drawings of Planar Graphs*

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Abstract. A natural way to draw two planar graphs whose vertex sets are matched is to assign each matched pair a unique y-coordinate. In this paper we introduce the concept of such matched drawings, which are a relaxation of simultaneous geometric embeddings with mapping. We study which classes of graphs allow matched drawings and show that (i) two 3-connected planar graphs or a 3-connected planar graph and a tree may not be matched drawable, while (ii) two trees or a planar graph and a planar graph of some special families—such as unlabeled level planar (ULP) graphs or the family of "carousel graphs"—are always matched drawable.

1 Introduction

The visual comparison of two graphs whose vertex sets are associated in some way requires drawings of these graphs that highlight their association in a clear manner. Drawings of this type are of use for various areas of computer science, including bio-informatics, web data mining, network analysis, and software engineering. Of course each drawing individually should be as clear as possible, using, for example, few bends and crossings. But, most importantly, the positions of associated vertices in the two drawings should be "close". This makes it possible for the user to easily identify structurally identical and structurally different portions of the two graphs, or to maintain her "mental map" [17]. Structural changes between two graphs and their visualizations arise, for example, when collapsing or expanding clusters in clustered drawings, during the navigation of very large graphs with a topological window, in the analysis of the evolving relationships among the actors of a social network, and in the comparison of multiple gene trees (see, for example, [1,6,7,11,14,16,18]).

Two positions are definitely "close" if they are identical. Hence a substantial research effort has recently been devoted to the problem of computing straight-line drawings of two graphs on the same set of points. More specifically, assume

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we are given two planar graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $|V_1| = |V_2|$, together with a one-to-one mapping between their vertices. A *simultaneous geometric embedding with mapping* (introduced by Brass et al. in [3]) of G_1 and G_2 is a pair of straight-line planar drawings Γ_1 and Γ_2 of G_1 and G_2, respectively, such that for any pair of matched vertices $u \in V_1$ and $v \in V_2$ the position of u in Γ_1 is the same as the position of v in Γ_2. Unfortunately, only pairs of graphs belonging to restricted subclasses of planar graphs admit a simultaneous geometric embedding with mapping. Brass et al. [3] showed how to simultaneously embed pairs of paths, pairs of cycles, and pairs of caterpillars, but they also proved that a path and a graph or two outerplanar graphs may not admit this type of drawing. Geyer, Kaufmann, and Vrt'o [15] recently proved that even a pair of trees may not have a simultaneous geometric embedding with mapping. These negative results motivated the study of relaxations of simultaneous geometric embeddings. One possibility is to introduce bends along the edges [4,8,9,13], another, to allow that the same vertex occupies different locations in the two drawings [2,3], introducing ambiguity in the mapping.

In this paper we consider a different interpretation of two positions being "close". Instead of requiring that matched vertices occupy the same location, we assign each matched pair a unique y-coordinate. This enables the user to unambiguously identify pairs of matched vertices but, at the same time, leaves us more freedom to draw both graphs clearly. Specifically, let again $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two planar graphs with $|V_1| = |V_2|$. G_1 and G_2 are *matched* if there is a one-to-one mapping between V_1 and V_2. If a vertex $u \in V_1$ is matched with a vertex $v \in V_2$ then we say that u is the *partner* of v and that v is the partner of u. A *matched drawing* of G_1 and G_2 is a pair of straight-line planar drawings Γ_1 and Γ_2 of G_1 and G_2, respectively, such that for any pair of matched vertices $u \in V_1$ and $v \in V_2$ the y-coordinate of u in Γ_1 is the same as the y-coordinate of v in Γ_2, and this y-coordinate is unique. If two matched graphs have a matched drawing, then we say that they are *matched drawable*. Matched drawings can be viewed as a relaxation of simultaneous geometric embedding with mapping. An example of a matched drawing of two trees is shown in Fig. 1.

Results and Organization. We start by presenting pairs of graphs that are not matched drawable. In particular, in Section 2.1 we describe two isomorphic 3-connected planar graphs that both have 12 vertices and that are not matched

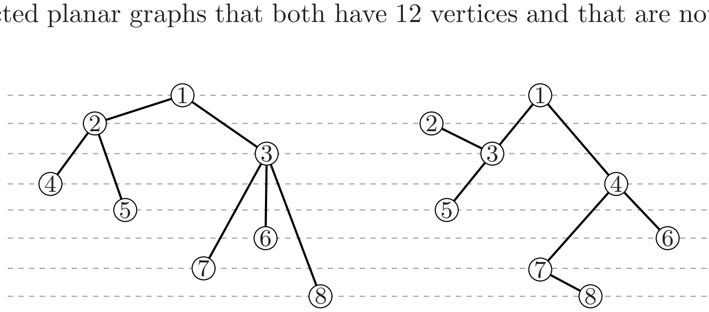

Fig. 1. A matched drawing of two trees

drawable. We also present a 3-connected planar graph and a tree that both have 620 vertices and that are not matched drawable. This construction can be found in Section 2.2.

We continue by describing drawing algorithms for classes of graphs that are always matched drawable. In particular, in Section 3.1 we show that a planar graph and an unlabeled level planar (ULP) graph that are matched are always matched drawable. In Section 3.2 we extend these results to a planar graph and a graph of the family of "carousel graphs". Finally, in Section 3.3 we prove that two matched trees are always matched drawable.

2 Graphs That Are Not Matched Drawable

2.1 Two 3-Connected Graphs

We start by stating a simple property of planar straight-line drawings.

Property 1. Let G be an embedded planar graph and let Γ be a straight-line planar drawing of G. Let u be the vertex of G with the highest y-coordinate in Γ and let v be the vertex of G with the lowest y-coordinate in Γ. Vertices u and v belong to the external face of G.

Now assume that G_1 and G_2 are two matched graphs with the following properties: (i) G_1 contains two vertex-disjoint simple cycles $C_1 = \{u_1, \dots, u_n\}$ and $C'_1 = \{u'_1, \dots, u'_m\}$, (ii) G_2 contains two vertex-disjoint simple cycles $C_2 = \{v_1, \dots, v_n\}$ and $C'_2 = \{v'_1, \dots, v'_m\}$, and (iii) u_i is the partner of v_i $(1 \leq i \leq n)$ and u'_j is the partner of v'_j $(1 \leq j \leq m)$. If Ψ_1 is a planar embedding of G_1 such that C'_1 is inside C_1 and if Ψ_2 is a planar embedding of G_2 such that C_2 is inside C'_2, then we call Ψ_1 and Ψ_2 *interlaced embeddings* and C_1, C'_1, C_2, and C'_2 *interlaced cycles.*

Lemma 1. *Let G_1 and G_2 be two matched graphs with interlaced embeddings Ψ_1 and Ψ_2. There is no matched drawing Γ_1 and Γ_2 of G_1 and G_2 such that Γ_1 preserves Ψ_1 and Γ_2 preserves Ψ_2.*

Proof. Denote by C_1, C'_1, C_2, and C'_2 the interlaced cycles of Ψ_1 and Ψ_2. Suppose by contradiction that Γ_1 and Γ_2 exist. Denote by $\overline{\Gamma_1}$ the subdrawing of Γ_1 restricted to the subgraph $C_1 \cup C'_1$ and by $\overline{\Gamma_2}$ the subdrawing of Γ_2 restricted to the subgraph $C_2 \cup C'_2$.

Since in Ψ_1 cycle C'_1 is inside cycle C_1, by Property 1 the top-most and the bottom-most vertices of $\overline{\Gamma_1}$ belong to C_1; denote these two vertices by u_t and u_b. Since $\overline{\Gamma_1}$ is planar and since the drawing of C'_1 is completely inside the drawing of C_1, every vertex u'_j of C'_1 has a y-coordinate that is greater than the y-coordinate of u_b and smaller than the y-coordinate of u_t. Since Γ_1 and Γ_2 are matched drawings, every vertex v'_j of C'_2 in $\overline{\Gamma_2}$ has a y-coordinate that is greater than the y-coordinate of v_b (i.e., the partner of u_b) and smaller than the y-coordinate of v_t (i.e., the partner of u_t). However, since in Ψ_2 cycle C_2 is inside cycle C'_2, by Property 1 the top-most and the bottom-most vertices of $\overline{\Gamma_2}$ belong to C'_2, a contradiction. □

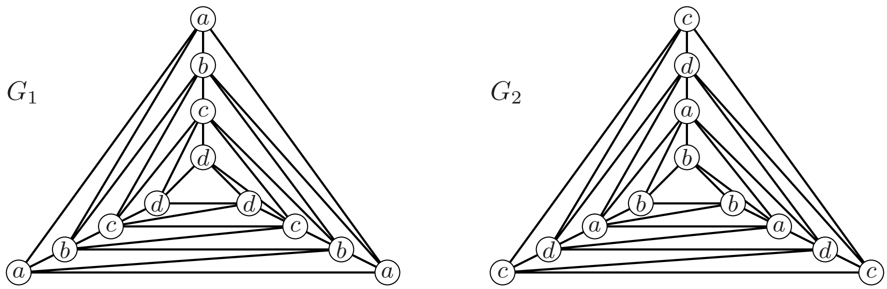

Fig. 2. Two 3-connected planar graphs that are not matched drawable. The partner of a vertex of G_1 is any vertex in G_2 that has the same label.

Theorem 1. *There exist two 3-connected planar graphs that are not matched drawable.*

Proof (sketch). Consider the two 3-connected planar graphs G_1 and G_2 in Fig. 2. The partner of a vertex of G_1 is any vertex in G_2 that has the same label. To prove that G_1 and G_2 are not matched drawable, we show that all planar embeddings of G_1 and G_2 are interlaced embeddings. The proof uses a case analysis on the choice of the external faces and is omitted for reasons of space. □

2.2 A 3-Connected Graph and a Tree

The two graphs described in Theorem 1 are both 3-connected. Hence the question arises if two planar graphs, at least one of which is not 3-connected, are always matched drawable. This is unfortunately not the case: in the following we present a planar graph and a tree that are not matched drawable.

Given a vertex v of a graph G and a drawing Γ of G, we denote by $x(v)$ and $y(v)$ the x- and y-coordinate of v in Γ. Let $T^* = (V^*, E^*)$ be the tree depicted in Fig. 3. Estrella-Balderrama et al. [10] proved the following lemma:

Lemma 2 (Estrella-Balderrama et al. [10]). *Let T^* be the tree depicted in Fig. 3. A straight-line planar drawing Γ of T^* such that $y(v_0) < y(v_7) < y(v_3) < y(v_2) < y(v_4) < y(v_1) < y(v_5) < y(v_6)$ in Γ does not exist.*

Let T^* be rooted at vertex v_0, and for each vertex v_i, denote by $d(v_i)$ the graph-theoretic distance of v_i from the root $(i = 0, 1, \ldots, 7)$. We construct a tree T by using T^* as a model. T has $3^{d(v_i)}$ copies of each vertex v_i $(i = 0, 1, \ldots, 7)$. The $3^{d(v_i)}$ copies of v_i are denoted as $v_{i,0}, v_{i,1}, \ldots, v_{i,3^{d(v_i)}-1}$. Vertex $v_{h,k}$ is a child of vertex $v_{i,j}$ in T if and only if v_h is a child of v_i in T^* and $j = \lfloor k/3 \rfloor$ $(0 \le i, h \le 7)$, $(0 \le j \le 3^{d(v_i)} - 1)$, $(0 \le k \le 3^{d(v_h)} - 1)$. So T has one copy of v_0 whose children are the three copies $v_{1,0}$, $v_{1,1}$, and $v_{1,2}$ of v_1. The children of each copy of v_1 are three of the nine copies of v_2, and so on. Three vertices of T with the same parent are called a *triplet* of T. The total number of vertices of T is 310.

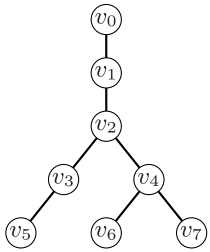

Fig. 3. A tree that does not have a straight-line planar drawing with $y(v_0) < y(v_7) < y(v_3) < y(v_2) < y(v_4) < y(v_1) < y(v_5) < y(v_6)$ [10]

Table 1. Matching between the vertices of T and the vertices of G_{103}

vertex	*copies*	*triplets*	*levels*
v_7	81	27	$1 \ldots 27$
v_3	27	9	$28 \ldots 36$
v_2	9	3	$37 \ldots 39$
v_4	27	9	$40 \ldots 48$
v_1	3	1	49
v_5	81	27	$50 \ldots 76$
v_6	81	27	$77 \ldots 103$

The tree T is matched with a *nested-triangles graph*, which is defined as follows. A single vertex v is a nested-triangles graph denoted by G_0. A triangulated planar embedded graph G_k $(k > 0)$ is a nested-triangles graph if the external face of G_k has exactly three vertices and the graph G_{k-1}, obtained by removing the vertices on the external face, is still a nested-triangles graph. A levelling of the vertices is naturally defined for the vertices of G_k: level i of G_k contains the vertices that are on the external face of G_i $(i = 0, 1, \ldots, k)$. Note that G_k has $3k + 1$ vertices and $k + 1$ levels. Thus, G_{103} has 310 vertices and 104 levels.

T and G_{103} are matched in the following way. Vertex v_0 is mapped to the (only) vertex of level 0. Each triplet of T is mapped to three vertices of G_{103} such that the level of these three vertices is the same in G_{103}. Also, all triplets formed by vertices that are copies of the same vertex of T^* are mapped to consecutive levels of G_{103}. The exact mapping is described in Table 1. Each row of the table refers to a different vertex of T^* and shows the number of copies of that vertex in T, the number of triplets in T, and the levels of G_{103} to which these triplets are mapped (a triplet for each level).

We now prove that, with the mapping described by Table 1, T and G_{103} are not matched drawable if we insist that the drawing of G_{103} preserves the embedding of G_{103}. We start with a useful property.

Property 2. Let $\Gamma_{G_{103}}$ be any planar straight-line drawing of G_{103} that preserves the embedding of G_{103}. For each level i $(0 \leq i \leq 103)$ there exists a vertex of level i that has y-coordinate greater than the y-coordinates of all the vertices having level less than i.

Lemma 3. *A matched drawing Γ_T and $\Gamma_{G_{103}}$ of the tree T and the graph G_{103} such that $\Gamma_{G_{103}}$ preserves the embedding of G_{103} does not exist.*

Proof (sketch). Let $\Gamma_{G_{103}}$ be any planar straight-line drawing of G_{103} that preserves the embedding of G_{103}. By exploiting Property 2, we can show that $\Gamma_{G_{103}}$ induces an ordering λ of the vertices of T along the y-direction such that there exists a subtree T' of T isomorphic to T^* for which the ordering λ restricted to the vertices of T' is the ordering given in Lemma 2 (the proof about how T' is defined is omitted). This implies that T' (and hence T) does not have a planar straight-line drawing that respects the ordering induced by $\Gamma_{G_{103}}$. □

According to Lemma 3, T and G_{103} are not matched drawable in the case that one wants a drawing of G_{103} that preserves the embedding of G_{103}. In the following theorem we show that T and G_{103} can be used to construct a new tree and a new 3-connected planar graph that are not matched drawable even if we allow the embedding to be changed.

Theorem 2. *There exist a tree and a 3-connected planar graph that are not matched drawable.*

Proof (sketch). Let $\overline{T}$ be a tree that consists of two copies of T whose roots are adjacent. Let G be a graph obtained by taking two distinct copies of G_{103} and connecting the vertices of their external faces in such a way that the obtained graph is a triangulated planar graph. The matching of the vertices is such that a copy of T matches a copy of G_{103} as before. We observe that any embedding of G leaves one of the copies of G_{103} as in Lemma 3. □

3 Matched Drawable Graphs

In this section we describe drawing algorithms for classes of graphs that are always matched drawable. In particular, in Section 3.1 we show that a planar graph and an unlabeled level planar (ULP) graph that are matched are always matched drawable. In Section 3.2 we extend these results to a planar graph and a graph of the family of "carousel graphs". Finally, in Section 3.3 we prove that two matched trees are always matched drawable.

These results show that matched drawings do indeed allow larger classes of graphs to be drawn than simultaneous geometric embeddings with mapping (a path and a planar graph may not admit a simultaneous geometric embedding with mapping [3] and the same negative result also holds for pairs of trees [15]).

3.1 Planar Graphs and ULP Graphs

ULP graphs were defined by Estrella-Balderrama, Fowler, and Kobourov [10]. Let G be a planar graph with n vertices. A *y-assignment* of the vertices of G is a one-to-one mapping $\lambda : V \rightarrow \mathbb{N}$. A *drawing of G compatible with λ* is a planar straight-line drawing of G such that $y(v) = \lambda(v)$ for each vertex $v \in V$. A planar graph G is *unlabeled level planar* (ULP) if for any given y-assignment λ of its vertices, G admits a drawing compatible with λ.

Theorem 3. *A planar graph and an ULP graph are always matched drawable.*

Proof (sketch). Let G_1 be a planar graph and let G_2 be an ULP graph. Compute a planar straight-line drawing of G_1 such that each vertex has a different y-coordinate, for example with a slight variant of the algorithm of de Fraysseix, Pach, and Pollack [5]. The drawing of G_1 together with the mapping between

G_1 and G_2 defines a y-assignment λ for G_2. Since G_2 is ULP it admits a drawing compatible with λ. It follows that G_1 and G_2 are matched drawable. □

ULP trees are characterized in [10]. A complete characterization of ULP graphs has very recently been given in [12]. A planar graph is ULP if and only if it is either a *generalized caterpillar*, or a *radius*-2 *star*, or a *generalized degree-3 spider*. These graphs are defined as follows (see also [12]). A graph is a *caterpillar* if deleting all vertices of degree one produces a path, which is called the *spine* of the caterpillar. A *generalized caterpillar* is a graph that contains cycles of length at most 4 in which every spanning tree is a caterpillar such that no three cut vertices are pairwise adjacent and no pair of adjacent cut vertices belong to the same 4-cycle. A *radius*-2 *star* is a $K_{1,k}$, $k > 2$, in which every edge is subdivided at most once. The only vertex of degree k is called the *center* of the star. A *degree*-3 *spider* is an arbitrary subdivision of $K_{1,3}$. A *generalized degree-3 spider* is a graph with maximum degree 3 in which every spanning tree is either a path or a degree-3 spider.

Corollary 1. *Let G_1 and G_2 be two matched graphs such that G_1 is a planar graph and G_2 is either a generalized caterpillar, or a radius-2 star, or a generalized degree-3 spider. Then G_1 and G_2 are matched drawable.*

3.2 Planar Graphs and Carousel Graphs

In this section we extend the result of Theorem 3 by describing a family of graphs that also includes non-ULP graphs and whose members have a matched drawing with any planar graph. Let G be a planar graph, let v be a vertex of G, and let Γ be a planar straight-line drawing of G. Γ is *v-stretchable* if: (i) there is a vertical ray from v going to $+\infty$ that does not intersect any edge of Γ, and (ii) for any given $\Delta > 0$, there exists a value $\Delta' \geq \Delta$ such that the drawing obtained by translating each vertex u with $x(u) \geq x(v)$ to point $(x(u) + \Delta', y(u))$ is still planar. Graph G is ULP *v-stretchable* if for every given y-assignment λ of its vertices, G admits a v-stretchable drawing compatible with λ.

A *carousel graph* is a connected planar graph G consisting of a vertex v_0, called the *pivot* of G, and of a set of disjoint subgraphs $S_1, \ldots, S_k$ ($k > 1$) such that each S_i has a single vertex v_i adjacent to v_0 ($i = 1, \ldots, k$) and S_i is ULP v_i-stretchable. Each subgraph S_i is called a *seat* of G. Vertex v_i is called the *hook* of S_i.

Theorem 4. *Any planar graph and any carousel graph that are matched are always matched drawable.*

Proof. Let G_1 be a planar graph and let G_2 by a carousel graph. Let v_0 be the pivot of G_2 and let u be the partner of v_0 in G_1. Compute a planar straight-line drawing of G_1 such that all vertices have different y-coordinates and u has the largest y-coordinate. The drawing of G_1 together with the mapping between G_1 and G_2 defines a y-assignment λ for G_2. Clearly $\lambda(w) < \lambda(v_0) = y_M$ for all vertices $w \neq v_0$ of G_2.

In the following we describe an incremental method to compute a drawing of G_2 compatible with λ. Let $S_1, \dots, S_k$ ($k > 1$) be the seats of G_2 and let v_i be the hook of S_i ($1 \le i \le k$). Let λ_i be the y-assignment of the vertices of S_i induced by λ. As a preliminary step we compute a drawing Γ_i for each S_i that is compatible with λ_i and that is v_i-stretchable. Such a drawing exists because S_i is ULP v_i-stretchable. We further assume that the distance between any two different x-coordinates is at least 1 unit.

We initialize the drawing by placing v_0 at position $(0, y_M)$, which results in drawing Γ_2^0. Drawing Γ_2^i is constructed from drawing Γ_2^{i-1} by adding drawing Γ_i at a suitable x-location and possibly translating some of its vertices further in x-direction (see Fig. 4). Hence the final drawing respects λ.

Let $\mathcal{R}_{i-1}$ be the bounding box of Γ_2^{i-1} and let (x_M, y_M) be the coordinates of its top-right corner. Further let R_i be the bounding box of Γ_i. Place the drawing Γ_i such that the left side of R_i is contained in the vertical line $x = x_M + 1$. Let R'_i be the (possibly empty) sub-rectangle of R_i delimited by the x-coordinates $x_M + 1$ and $x'_M = x(v_i) - 1$. Further let y'_M denote the maximum y-coordinate of any vertex of Γ_2^{i-1} or Γ_i different from v_0 and let $p = (x'_M + 1, y'_M)$. The line ℓ through v_0 and p crosses neither Γ_2^{i-1} nor the portion of Γ_i contained in R'_i (see Fig. 4(a)). Let q denote the intersection of ℓ with the horizontal line at $y(v_i)$ and let $\Delta = x(q) - x(v_i)$. Since Γ_i is v_i-stretchable, there exists a value $\Delta' \ge \Delta$ such that we can translate the portion of Γ_i contained in $R_i \setminus R'_i$ to the right by Δ' without creating any crossing (see Fig. 4(b)). It can easily be verified that we can now connect v_i to v_0 without creating any crossings. □

Lemma 4. *Let G be a simple cycle and let v be any vertex of G. G is ULP v-stretchable.*

Proof. Let λ be any y-assignment of the vertices of G and let u be the vertex of G that has the smallest y-coordinate. Let $u = v_0, v_1, \dots, v_{n-1}$ be the vertices of G in the order they are encountered when walking clockwise along G. Place each vertex v_i at point $(i, \lambda(v_i))$. Clearly none of the edges (v_i, v_{i+1}) ($i = 0, 1, \dots, n-2$) cross each other. To avoid crossings between edge (v_0, v_{n-1}) and the other edges we translate v_{n-1} to the right until the segment connecting v_0 to v_{n-1} does not cross any other segment. It is immediate to see that such a drawing is v-stretchable for every vertex v of G. □

Corollary 2. *Let G_1 and G_2 be two matched graphs such that G_1 is a planar graph and G_2 is a cycle. Then G_1 and G_2 are matched drawable.*

The drawing techniques in [10] imply the following two lemmata.

Lemma 5. *Let G be a caterpillar and let v be a vertex of its spine. G is ULP v-stretchable.*

Lemma 6. *Let G be a radius-2 star and let v be the center of G. G is ULP v-stretchable.*

Corollary 3. *Let G_1 and G_2 be two matched graphs such that G_1 is a planar graph and G_2 is a carousel graph. If each seat of G_2 is either a caterpillar with*

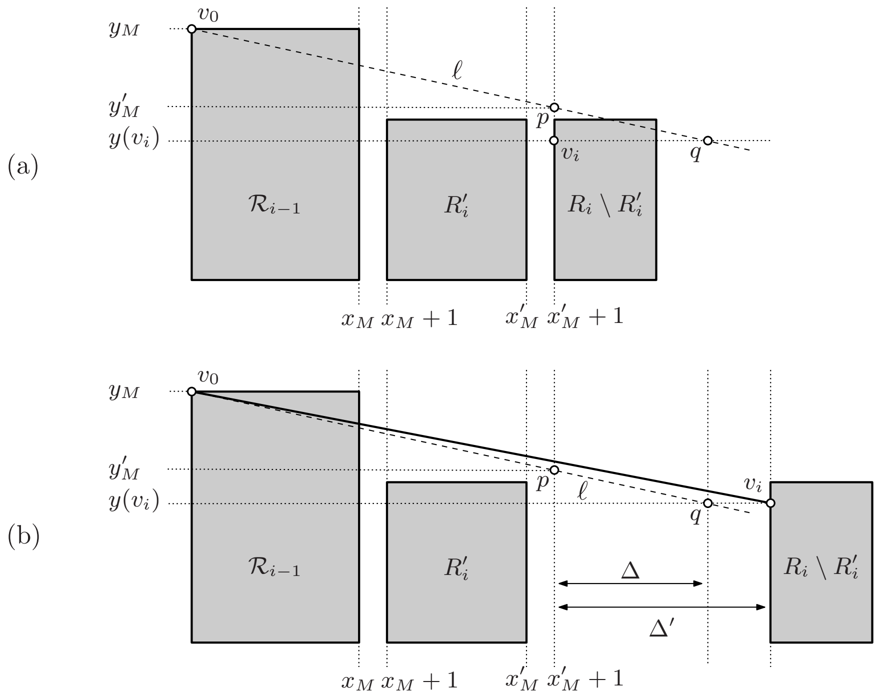

Fig. 4. Adding Γ_i to Γ_2^{i-1}

*a vertex of its spine as its hook, a radius-*2 *star with its center as its hook, or a cycle, then* G_1 *and* G_2 *are matched drawable.*

The family of carousel graphs described by Corollary 3 contains graphs that are not ULP. For example, the graph depicted in Fig. 3 is a carousel graph with pivot v_2, the three seats are caterpillars.

3.3 Two Trees

Theorem 5. *Any two matched trees are matched drawable.*

Proof. Let T_1 and T_2 any two matched trees. We prove by construction that T_1 and T_2 are matched drawable. Let the y-coordinates to be used be $1, \ldots, n$, we will assign matched vertices from T_1 and T_2 consecutively to coordinates $n, 1, n-1, 2, n-2, 3, \ldots$ until all vertices are placed.

Let T_i be a tree with a subset of its vertices placed. Then the maximal connected unplaced parts of T_i are incident to one, two, or more placed vertices. We call a maximal connected unplaced part of a tree a *chunk*.

We maintain the following invariant for T_1: after every odd placement, every chunk of T_1 is incident to at most two placed vertices of T_1. For T_2 we maintain a similar invariant: after every even placement, every chunk of T_2 is incident to at

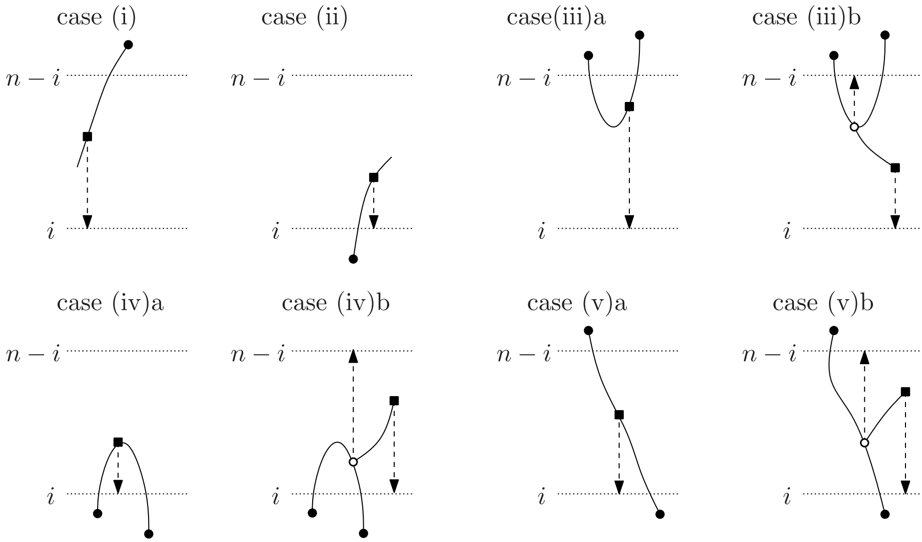

Fig. 5. The eight cases for placement at i (and at $n-i$ in three cases)

most two placed vertices of T_2. We call this the *topological invariant*. Intuitively, tree T_1 determines which vertex is placed in odd placements at n, $n-1$, $n-2$, ..., and tree T_2 determines which vertex is placed in even placements at 1, 2, 3, ... The other tree just follows with the matched vertex.

The topological invariants are needed for two reasons. Most importantly, they make sure that the algorithm cannot get stuck, in the sense that the placement of a vertex leads to an intersection. Secondly, they limit the number of cases that must be analyzed.

Consider T_1 after an odd placement and assume that it satisfies the invariant. Then a chunk can be one of five *types*: (1) it has one incident placed vertex at a high coordinate; (2) it has one incident placed vertex at a low coordinate; (3) it has two incident placed vertices at high coordinates; (4) it has two incident placed vertices at low coordinates; (5) it has one incident placed vertex at a high coordinate and one incident placed vertex at a low coordinate.

An even placement (at the bottom) may cause violation of the invariant for T_1 unless the next odd placement restores it. So for the case analysis of T_1 we will consider all possibilities of an even placement and the corresponding odd placement. If the even placement is at i, then the next odd placement is at $n-i$. There are eight cases to be distinguished for an even placement at i; they are shown in Fig. 5. In the three (.)b cases, which vertex to place at $n-i$ is determined by the fact that the topological invariant must be restored for T_1. It is the unique vertex of T_1 where the path from the just placed vertex meets the path between the two vertices that bound the chunk. It is easy to see from the figure that in the three (.)b cases the invariant can be restored for T_1 by

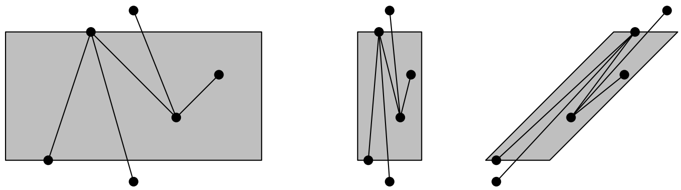

Fig. 6. Scaling and shearing a wide rectangle into a narrow parallelogram

placing this vertex at $n - i$. In the five other cases, we can assure the invariant to hold after placement at $n - i$ by choosing to place any unplaced vertex that is a neighbor of a placed vertex.

The situation is completely analogous for T_2, where an odd placement may cause a violation of the invariant if the next even placement is not chosen well.

Next we must show that there is actually space to draw the trees without crossings and with straight edges. For this we need a geometric invariant: after the placement at $n - i + 1$, there is a parallelogram between the horizontal lines at $n - i$ and i in which the whole chunk can be drawn without crossings and with straight edges. The parallelograms must have positive width and have an "alignment" that corresponds to the needs of the chunk. For example, for type (1) the incident placed vertex must be able to connect to any point on the far horizontal side of the parallelogram without going outside the parallelogram. It remains to show that every chunk can be drawn inside its parallelogram and that, if a chunk is split into several chunks, their resulting parallelograms are disjoint. In essence this is the case because any parallelogram can be scaled and sheared to fit, see Fig. 6. The formal statement of the geometric invariant and the remainder of the proof are omitted due to space limitations. □

4 Conclusions and Open Problems

In this paper we introduced the concept of matched drawings, which are a natural way to draw two planar graphs whose vertex sets are matched. Since this is the first study of these drawings, many interesting and challenging open problems remain. First of all, in the light of Theorems 2 and 4, we would like to characterize the subclass of planar graphs that admit a matched drawing with any planar graph. Secondly, the drawing techniques of Theorems 4 and 5 may give rise to drawings where the area is exponential in the size of the graphs. It would be interesting to study the area requirement of matched drawings that use straight-line edges. On a related note, some of our drawing techniques rely on a planar straight-line drawing of a planar graph where each vertex has a different y-coordinate. How big a grid is necessary to guarantee such a drawing with integer coordinates? And finally, given any two matched graphs, what is the complexity of testing whether they are matched drawable?

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